Documento de Trabajo 9610

# AN APROXIMATION TO THE VARIANCE OF AN INTEGRAL ESTIMATOR OF THE RETRIAL PARAMETER IN THE M/G/1 RETRIAL QUEUE

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# AN APPROXIMATION TO THE VARIANCE OF AN INTEGRAL ESTIMATOR OF THE RETRIAL PARAMETER IN THE M/G/1 RETRIAL QUEUE

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**RESUMEN**

En este artículo se obtiene una expresión explícita de la varianza asintótica de un estimador integral del parámetro de reintento en un sistema de colas *M/G/*1 con reintentos. Dicha expresión se obtiene resolviendo ciertas ecuaciones diferenciales y depende de los momentos factoriales del proceso $(M(t),N(t))$, donde $M(t)$ es el número total de llegadas desde que se produce la última salida hasta el instante $t$ y $N(t)$ el número de individuos en órbita en el instante $t$.

**ABSTRACT**

In this article we obtain an explicit expression for the asymptotic variance of an integral estimator of the retrial parameter in the *M/G/*1 retrial queue. This expression is obtained by solving some linear differential equations and it depends on the factorial moments of the stochastic process $(M(t),N(t))$ where $M(t)$ is the total number of arrivals from the last departure until time $t$ and $N(t)$ is the number of customers in orbit at time $t$.

**KEYWORDS**

Queues, Repeated Attempts, Statistical Inference.

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## INTRODUCTION.

We consider a queueing system characterized by the fact that when a customer arrives into the system and finds the channel busy he leaves the service area and applies again for service after a random period. We say that these customers are "in orbit". On the other hand if an arriving customer finds the channel free he occupies it and leaves the system after his service completion (these customers are identified as primary calls). A system under this description is called a Retrial Queue. Such queueing systems are used as models in telecommunication networks where the phenomena of repeated attempts must be taken into account.

In the *M/G/*1 retrial queue primary calls arrive according to a Poisson process at the rate $\lambda$ and each customer in orbit reapplies for service according to an independent Poisson distribution at an unknown rate $\mu$. The service time distribution function, $B(x)$, is assumed to be the same for both primary and repeated calls. The input flow of primary calls, the intervals between repetitions and the service time are taken to be mutually independent. A review of the main results and other aspects of the literature on this topic can be found in Yang and Templeton (1987) and in Falin (1990).

The M/G/1 retrial queue has been studied by Keilson, Cozzolino and Young (1968) using the Markovian process $(C(t),N(t),\xi(t))$ where $C(t)$ represents the number of busy channels (for single-channel queues $C(t)$=1 or 0 according to whether the channel is busy or free), $N(t)$ is the number of subscribers whose previous attempts to get service were unsuccessful and try to get service again and the supplementary variable $\xi(t)$, when $C(t)$=1, denotes the time elapsed while the call is being served.

When trying to estimate the parameter of retrial this description is insufficient because only effective arrivals (i.e. primary calls and repeated calls who find the service free) are taken into account. Moreover, we cannot distinguish between primary and repeated calls and the number of customers in orbit is usually unobserved. Hence, the estimation of the retrial parameter becomes a difficult problem (see for example Lewis and Leonard (1982), Warlfield and Foers (1985) and Hoffman and Harris (1986)).

Recently, G. Falin (1995) introduced, for the M/M/1 case, another Markovian characterization of single server retrial queues. He suggested the replacement of the component $C(t)$ by the process $M(t)$ which describes the total number of attempts to get service since the last departure time. This process was found to be extremely useful in the estimation of the retrial rate.

This new characterization of the $M/G/1$ retrial queues is investigated in Rodrigo et al.(1996). In that paper is also analysed the estimation problem of the retrial parameter, $\mu$, using the estimator

$$\zeta_T = a\alpha_T - b, \tag{1}$$

where $T$ is the observation time,

$$\alpha_T = \frac{1}{T}\int_0^T M(t)dt \tag{2}$$

and $a$ and $b$ are known constants given by

$$a = \frac{12(1-\rho)}{2\lambda^2(1-\rho)\beta_3 + 3\lambda^3\beta_2^2}, \qquad b = \frac{6\lambda^2\beta_2 + 12\rho(1-\rho)}{2\lambda^2(1-\rho)\beta_3 + 3\lambda^3\beta_2^2}. \tag{3}$$

($\beta_i$ is the $i$-th central moment of the service time and $\rho=\lambda\beta_1$ is the traffic intensity).

In the present article we obtain, in full detail, the explicit expression for the asymptotic variance of $\zeta_T$. In section one we consider the case when $\lambda$ is known. The case of $\lambda$ unknown is analyzed in section two.

Throughout the paper we will assume that the stochastic processes $C(t)$, $M(t)$, $N(t)$ and $\xi(t)$ are in steady state but retain the $t$ in the notation since it will be convenient for later use.

## 1. VARIANCE APPROACH ($\lambda$ known).

The statistical accuracy of $\zeta_T$ can be measured by $Var[\zeta_T]=a^2Var[\alpha_T]$. To obtain an asymptotic expression for $Var[\alpha_T]$, we use the method suggested in Falin (1996) for the case where both $\mu$ and $\lambda$ are unknown. In this section we restrict ourselves to the case where only $\mu$ is unknown. Extensions to the case where both $\lambda$ and $\mu$ are unknown are considered in section two. In any case, observe that $\lambda$ can be estimated using the interdeparture times, since the mean interdeparture time in steady state is equal to $\lambda$ for ergodic *M/G/1* retrial queues.

We will prove that, under certain conditions,

$$Var[\alpha_T]=\frac{2}{T}V + o(1/T)$$

where $V$ is a constant which depends on $M\equiv E[M(t)]$, $\lambda$ and $\beta_i$, $i=1,\dots,6$ (see theorem-3). This constant can be written as a function of the factorial moments calculated in Rodrigo et al. (1996). The mechanism used to get $V$ is not difficult, but it requires tedious algebra. Using equation (17) in Falin (1995), we can write

$$Var[\alpha_T] = \frac{2}{T^2}\int_0^T (T-t)R(t)dt \tag{4}$$

where $R(t)=E[Y(t)Y(0)]-E[Y(t)]E[Y(0)]$ is the covariance function of the stationary process $Y(t)=(M(t),N(t),\xi(t))$. In order to get an asymptotic expression for $Var[\alpha_T]$ we use the transition probabilities of the stationary Markovian process $Y(t)$, which are denoted by

$$p_{mn}(t;x|m',n')dx=\Pr[M(t)=m, N(t)=n, x\le\xi(t)<x+dx|M(0)=m',N(0)=n'],$$
$$m\ge1,\quad t>0$$

when the service is busy ($x=0$ represents the instant where service starts and therefore $m=1$), and by

$$p_{0n}(t|m',n')=\Pr[M(t)=0,N(t)=n|M(0)=m',N(0)=n'],\quad t>0$$

when the service is free at time $t$. At time $t=0$ we have

$$p_{0n}(0|m',n')=\delta((0,n),(m',n'))$$

and

$$p_{mn}(0;x|m',n')p_{m'n'}=\delta((m,n),(m',n'))p_{mn}(x),\quad m\ge1,$$

where $p_{m'n'}$ and $p_{mn}(x)$ are the stationary probabilities of the processes $(M(t),N(t))$ and $Y(t)$ respectively, and $\delta((a,b),(a',b'))=1$ if $a=a'$ and $b=b'$, and is zero otherwise. Due to the Markovian character of the process $Y(t)$, we can write the covariance function as

$$R(t) = \int_0^\infty R(t;x)dx \tag{5}$$

where

$$R(t;x)=\sum_{m=1}^{\infty} m\sum_{n=0}^{\infty}\ \sum_{m'=1}^{\infty} m'\sum_{n'=0}^{\infty} p_{m'n'}(p_{mn}(t;x|m',n')-p_{mn}(x)).$$

Then, using Fubini's theorem, we can write (4) as

$$Var[\alpha_T] = \int_0^\infty V_T(x)dx$$

where

$$V_T(x)=\frac{2}{T^2}\int_0^T (T-t)R(t;x)dt.$$

Moreover, if $Y(t)$ ($\xi(t)$=x fixed) is an uniformly geometrically ergodic process, we have that

$$V_T(x)=\frac{2}{T}\int_0^\infty R(t;x)dt + o_x(1/T).$$

(see Falin (1996), lemma 1), where $o_x(1/T)$ has finite integral in the variable $x$, $x\in(0,\infty)$, since $R(t)$ exists (see (5)). Therefore

$$Var[\alpha_T]\approx\frac{2}{T}V + o(1/T) \qquad (6)$$

where

$$V=\int_0^\infty\int_0^\infty R(t;x)dtdx.$$

We will next use some transformations of the Chapman-Kolmogorov equations and we will obtain $V$ after solving some linear differential equations. The Chapman-Kolmogorov equations expresed in terms of transition probabilities are

$$\left.\begin{aligned}
&\frac{dp_{0n}(t|m',n')}{dt} = -(\lambda+n\mu)p_{0n}(t|m',n') + \int_0^\infty\left(\sum_{m=1}^{\infty} p_{mn}(t;x|m',n')\right)b(x)dx\\
&\frac{\partial p_{mn}(t;x|m',n')}{\partial t} + \frac{\partial p_{mn}(t;x|m',n')}{\partial x} = -(\lambda+n\mu+b(x))p_{mn}(t;x|m',n') +\\
&\qquad + \left[\lambda p_{m-1,n-1}(t;x|m',n') + n\mu p_{m-1,n}(t;x|m',n')\right](1-\delta_{1m})\\
&p_{1n}(t;0|m',n') = \lambda p_{0n}(t|m',n') + (n+1)\mu p_{0,n+1}(t|m',n')\\
&p_{mn}(t;0|m',n') = 0, \quad \text{if } m\geq 2
\end{aligned}\right] \quad (7)$$

After calculating the Laplace transform in the variable $t$, these equations become

$$\left.\begin{aligned}
&-\delta((0,n),(m',n'))+sp^*_{0n}(s|m',n') = -(\lambda+n\mu)p^*_{0n}(s|m',n') +\\
&\qquad + \int_0^\infty \sum_{m=1}^{\infty} p^*_{mn}(s;x|m',n')b(x)dx\\
&-p_{mn}(0,x|m',n')+sp^*_{mn}(s;x|m',n')+\frac{\partial p^*_{mn}(s;x|m',n')}{\partial x} =\\
&\quad = \left[\lambda p^*_{m-1,n-1}(s;x|m',n')+n\mu p^*_{m-1,n}(s;x|m',n')\right](1-\delta_{1m}) -\\
&\quad - (\lambda+n\mu+b(x))p^*_{mn}(s;x|m',n'), \quad m\geq 1\\
&p^*_{1n}(s;0|m',n') = \lambda p^*_{0n}(s|m',n') + (n+1)\mu p^*_{0,n+1}(s|m',n')\\
&p^*_{mn}(s;0|m',n') = 0, \quad m\geq 2
\end{aligned}\right] \quad (8)$$

where $p^*_{..}(s;.|m',n')$ are the corresponding Laplace transforms. Dividing by s the stationary Chapman-Kolmogorov equations given by

$$\left.\begin{aligned}&(\lambda+n\mu)p_{0n}=\sum_{m=1}^{\infty}\int_0^{\infty}p_{mn}(x)b(x)dx\\&p'_{mn}(x)=-(\lambda+n\mu+b(x))p_{mn}(x)+(1-\delta_{1m})(\lambda p_{m-1,n-1}(x)+n\mu p_{m-1,n}(x))\end{aligned}\right\}$$

where $p_{0n}$ and $p_{mn}(x)$ are the statiorary versions of $p_{0n}(t|m',n')$and $p_{mn}(t;x|m',n')$ respectively (see Rodrigo et al. (1996)), and subtracting them from the analogous equations in (8) and taking the limit when $s\rightarrow 0$, we get

$$\left.\begin{aligned}&-\delta((0,n),(m',n'))+p_{0n}=-(\lambda+n\mu)V_{0n}(m',n')+\int_0^{\infty}\sum_{m=1}^{\infty}V_{mn}(x|m',n')b(x)dx\\&-p_{mn}(0,x|m',n')+p_{mn}(x)+\frac{dV_{mn}(x|m',n')}{dx}=\\&\qquad=\Big[\lambda V_{m-1,n-1}(x|m',n')+n\mu V_{m-1,n}(x|m',n')\Big](1-\delta_{1m})-\\&\qquad-(\lambda+n\mu+b(x))V_{mn}(x|m',n'),\quad m\geq 1\\&V_{1n}(0|m',n')=\lambda V_{0n}(m',n')+(n+1)\mu V_{0,n+1}(m',n')\\&V_{mn}(0|m',n')=0,\quad m\geq 2\end{aligned}\right]\qquad(9)$$

where

$$V_{0n}(m',n')=\int_0^{\infty}\Big(p_{0n}(t|m',n')-p_{0n}\Big)dt=\lim_{s\to 0}\Big(p^*_{0n}(s|m',n')-\frac{1}{s}p_{0n}\Big)$$

$$\begin{aligned}V_{mn}(x|m',n')&=\int_0^{\infty}\Big(p_{mn}(t,x|m',n')-p_{mn}(x)\Big)dt=\\&=\lim_{s\to 0}\Big(p^*_{mn}(s,x|m',n')-\frac{1}{s}p_{mn}(x)\Big),\quad m\geq 1.\end{aligned}$$

Multiplying each equation in (9) by $m'p_{m'n'}$, and adding up the variables $m'$ and $n'$ $(m'\geq 1,\ n'\geq 0)$ we have

$$\left.\begin{aligned}
&Mp_{0n}=-(\lambda+n\mu)V_{0n}+\int_0^{\infty}\sum_{m=1}^{\infty}V_{mn}(x)b(x)dx\\
&Mp_{mn}(x)-mp_{mn}(x)+\frac{dV_{mn}(x)}{dx}=-(\lambda+n\mu+b(x))V_{mn}(x)+\\
&\qquad\qquad+\left[\lambda V_{m-1,n-1}(x)+n\mu V_{m-1,n}(x)\right](1-\delta_{1m})\\
&V_{1n}(0)=\lambda V_{0n}+(n+1)\mu V_{0,n+1}\\
&V_{mn}(0)=0,\quad m\geq 2
\end{aligned}\right] \tag{10}$$

where $M=E[M(t)]$ and

$$V_{0n}=\sum_{m'=1}^{\infty}m'\sum_{n'=0}^{\infty}p_{m'n'}V_{0n}(m',n')$$

$$V_{mn}(x)=\sum_{m'=1}^{\infty}m'\sum_{n'=0}^{\infty}p_{m'n'}V_{mn}(x|m',n'),\quad m\geq 1.$$

Note that $V_{0n}$ and $V_{mn}(x)$, $m\geq 1$, satisfy the normalizing condition

$$\sum_{n=0}^{\infty}V_{0n}+\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\int_0^{\infty}V_{mn}(x)dx = 0. \tag{11}$$

With this new notation the constant $V$ is given by

$$\int_0^{\infty}\left(\sum_{m=1}^{\infty}m\sum_{n=0}^{\infty}V_{mn}(x)\right)dx. \tag{12}$$

To obtain this constant, we use the generating functions given in (10) and (11) and get

$$MP_0(z) = -\lambda V_0(z) - \mu z V_0'(z) + \int_0^\infty V(1,z;x)b(x)dx \tag{13}$$

$$\begin{aligned} MH(y,z;x) - yH_y'(y,z;x) &= (\lambda yz - \lambda - b(x))V(y,z;x) + \\ &+ \mu z(y-1)V_z'(y,z;x) - \frac{dV(y,z;x)}{dx} \end{aligned} \tag{14}$$

$$V(y,z;0) = \lambda y V_0(z) + \mu y V_0'(z) \tag{15}$$

$$V_0(1) + \int_0^\infty V(1,1;x)dx = 0 \tag{16}$$

where

$$P_0(z) = \sum_{n=0}^{\infty} p_{0n} z^n$$

$$V_0(z) = \sum_{n=0}^{\infty} z^n V_{0n}, \qquad V(y,z;x) = \sum_{m=1}^{\infty} y^m \sum_{n=0}^{\infty} z^n V_{mn}(x)$$

and where the prime denotes derivatives (we write a subindex to indicate the variable with respect to which we are differentiating, except for $P_0(z)$ and $V_0(z)$ where we always differentiate with respect to $z$). From formula (12), the constant $V$ can be written as

$$V = \int_0^\infty V_y'(1,1;x)dx. \tag{17}$$

The function $V_y'(1,1;x)$ can be obtained using equations (13), (14), (15) and (16). To simplify our calculations we introduce the following notation. Let

$$\vartheta(y,z;x) \equiv MH(y,z;x) - yH_y'(y,z;x), \qquad \vartheta(x) \equiv \vartheta(1,1;x),$$

$$q(x) \equiv \vartheta(x)/(1-B(x)).$$

Let the integral operators be

$$I(\vartheta(x)) \equiv I^1(\vartheta(x)) \equiv \int q(x)dx, \qquad I^n(\vartheta(x)) = \int I^{n-1}(\vartheta(x))dx, \quad n \geq 2.$$

These are polynomials in the variable $x$ of degree $n$, $n \geq 1$, and independent term equal to zero. Finally let

$$I^n(\vartheta(\beta)) \equiv I^n(\vartheta(x))\Big|_{x^j=\beta_j} \quad ; \quad 1 \leq j \leq n$$

An explicit formula for $V'_y(1,1,x)$ depends on terms which we obtain in the following lemmas.

**Lemma 1**

a) $V(1,1;x) = [V(1,1;0) - I(\vartheta(x))](1-B(x))$

b) $V'_z(1,1;x) = [V'_z(1,1;0) + \lambda V(1,1;0)x - \lambda I^2(\vartheta(x)) - I(\vartheta'_z(x))](1-B(x))$

c) $V''_{zz}(1,1;x) = [V''_{zz}(1,1;0) + 2\lambda V'_z(1,1;0)x + \lambda^2 V(1,1;0)x^2 - 2\lambda^2 I^3(\vartheta(x)) -$
$- 2\lambda I^2(\vartheta'_z(x)) - I(\vartheta''_{zz}(x))](1-B(x))$

*Proof.* (a) Set z=y=1 in (14) and solve. (b) Differentiating in (14) with respect to z at z=y=1 and solve. (c) Differentiating twice in (14) with respect to z at z=y=1 and solve.

■

The functions $V(1,1;x)$, $V'_z(1,1;x)$ and $V''_{zz}(1,1;x)$ depend on the constants $V(1,1;0)$, $V'_z(1,1;0)$ and $V''_{zz}(1,1;0)$ respectively. Expressions for the first two constants are obtained in the following lemma. We do not need the constant $V''_{zz}(1,1;0)$.

**Lemma 2**

a) $V(1,1;0) = V'_y(1,1;0)=\lambda A-B$

b) $V'_z(1,1;0) = (1-\rho)^{-1}[\lambda(2^{-1}\lambda\beta_2+\mu^{-1}\rho)(\lambda A-B)-\lambda\mu^{-1}B-2^{-1}C]$, where

$$A=I^2(\vartheta(\beta)), \qquad B=MP'_0(1)+\lambda I^2(\vartheta(\beta))+I(\vartheta'_z(\beta))$$

and

$$C=MP''_0(1)+2\lambda^2I^3(\vartheta(\beta))+2\lambda I^2(\vartheta'_z(\beta))+I(\vartheta''_{zz}(\beta)).$$

*Proof.* Setting $z=y=1$ and differentiating in (15) at that point with respect to $y$, $z$ and $zz$ we obtain

$$V(1,1;0)=V'_y(1,1;0)=\lambda V_0(1)+\mu V'_0(1) \quad (18)$$

$$V'_z(1,1;0)=\lambda V'_0(1)+\mu V''_0(1) \tag{19}$$

$$V''_{zz}(1,1;0)=\lambda V''_0(1)+\mu V'''_0(1). \tag{20}$$

By lemma 1, equation (16) becomes

$$V_0(1)+V(1,1;0)\beta_1=A. \tag{21}$$

To find $V(1,1;0)$, $V'_z(1,1;0)$ and $V'_y(1,1;0)$ we need some more work. Differentiating (13) with respect to $z$ at $z=1$ we have

$$MP'_0(1)=-\lambda V'_0(1)-\mu V'_0(1)-\mu V''_0(1)+\int_0^\infty V'_z(1,1;x)b(x)dx \tag{22}$$

and using lemma 1(b) together with (19) gives

$$\rho V(1,1;0)-\mu V'_0(1)=B. \tag{23}$$

Using (18), (19) and (23) we get $V(1,1;0)=V'_y(1,1;0)=\lambda A-B$. Differentiating (13) twice with respect to $z$ at $z=1$ we have

$$MP''_0(1)=-\lambda V''_0(1)-2\mu V''_0(1)-\mu V'''_0(1)+\int_0^\infty V''_{zz}(1,1;x)b(x)dx. \tag{24}$$

Finally using lemma 1(c) and equations (20), (22) and (24) we calculate $V'_z(1,1;0)$. ■

**Theorem 3**

*Assume* that *an ergodic M/G/1 retrial queue, with retrials exponentially distributed,* is *observed during the interval* $(0,T]$. *Let* $B(x)$, $x\geq 0$, *be the service time distribution with* $\beta_i<\infty$, $i=1,\dots,6$. *Then, the variance of the estimator* $\zeta_T=a\alpha_T+b$, *where* $a$, $b$ *and* $\alpha_T$ *are given in* (19) *and* (20), *is*

$$Var(\zeta_T)=\frac{2V}{T}\left(\frac{12(1-\rho)}{2\lambda^2(1-\rho)\beta_3+3\lambda^3\beta_2^2}\right)^2+o(1/T) \tag{25}$$

*where*

$$V=\left(\beta_1+\frac{\lambda\beta_2}{2}+\frac{\lambda\mu\beta_3}{6}\right)V(1,1;0)+\frac{\lambda\beta_2}{2}V'_z(1,1;0)-\lambda\mu I^4(\vartheta(\beta))-$$

$$-\lambda I^3(\vartheta(\beta))-\mu I^3(\vartheta'_z(\beta))-I^2(\vartheta'_y(\beta)). \tag{26}$$

*Proof*. Differentiating (14) with respect to $y$ at $z=y=1$ we get

$$\frac{d}{dx}V'_y(1,1;x)=-b(x)V'_y(1,1;x)+\lambda V(1,1;x)+\mu V'_z(1,1;x)-\vartheta'_y(x)$$

whose solution is (see lemmas 1 and 2)

$$V'_y(1,1;x)=[\lambda I(V(1,1;x))+\mu I(V'_z(1,1;x))-I(\vartheta'_y(x))+V'_y(1,1;0)](1-B(x))$$

and the theorem follows from (17). ■

**Remark.** Note that

$$\vartheta(x)=MF(0,0;x)-F(1,0;x), \quad \vartheta'_z(x)=MF(0,1;x)-F(1,1;x)$$

$$\vartheta''_{zz}(x)=MF(0,2;x)-F(1,2;x), \quad \vartheta'_y(x)=(M-1)F(1,0;x)-F(2,0;x)$$

where $F(i,j;x)$ is the $(i,j)$-th factorial moment of $(N(t),M(t),\xi(t)=x))$ and

$$I^kF(i,j;\beta)=\sum_{n=0}^{2i+1} G_n(i,j)\,\frac{n!}{(n+k)!}\,\beta_{n+k}$$

where $G_n(i,j)$ are coefficients which can be calculated recurrently (see Rodrigo et al. (1996)). Hence using the factorial moments we could write an explicit formula for $Var(\zeta_T)$. ■

## 2. VARIANCE APPROACH ($\lambda$ unknown).

We now consider the estimator

$$\omega_T=\frac{1}{T\beta_1}\int_0^\infty C(t)dt \tag{27}$$

for the parameter $\lambda$. Then, the estimator $\zeta_T$ is given by

$$\zeta_T=a(\omega_T)\alpha_T-b(\omega_T)$$

where $a(\omega_T)$ and $b(\omega_T)$ are the constants given in (3) where $\omega_T$ takes the place of $\lambda$. We can approximate the variance of $\zeta_T$ (see Falin (1996)) by

$$Var[\zeta_T] \simeq \{a'(\lambda)M-b'(\lambda)\}^2 Var[\omega_T] + 2M\{a'(\lambda)M-b'(\lambda)\}Cov[\omega_T,\alpha_T] +$$
$$+ (a'(\lambda))^2 Var[\alpha_T].$$

To calculate $Var[\omega_T]$ and $Cov[\omega_T,\alpha_T]$ we can proceed as we did with the variance $Var[\alpha_T]$. In that case we get

$$Var[\omega_T]=-\frac{2}{T}W_0(1)$$

$$Cov[\omega_T,\alpha_T]=\frac{1}{T}[W-V_0(1)]$$

where $V_0(1)=A(1-\rho)+B\beta_1$ is obtained from equation (21) and the constants $W$ and $W_0(1)$ are the analogous of $V$ and $V_0(1)$ where $\rho P_0(z)$ and $(\rho-1)H(y,z;x)$ take the place of $MP_0(z)$ and $MH(y,z;x)-yH'_y(y,z;x)$ in equations (13) and (14) respectively (details are omitted). Lemmas 1 and 2 and theorem 3 are also valid for the variables $W$. The function $\vartheta(y,z;x)$ is equal to $(\rho-1)H(y,z;x)$ in this case.

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